Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **73**, 1-7; **http://www.math.u-szeged.hu/ejqtde/**

EXISTENCE OF GLOBAL SOLUTIONS FOR A SYSTEM OF REACTION-DIFFUSION EQUATIONS WITH EXPONENTIAL NONLINEARITY

EL HACHEMI DADDIOUAISSA

Abstract. We consider the question of global existence and uniform boundedness of nonnegative solutions of a system of reaction-diffusion equations with exponential nonlinearity using Lyapunov function techniques.

1. Introduction

In this paper we consider the following reaction−diffusion system

$$\frac{\partial u}{\partial t} - a\Delta u = \Pi - f(u,v) - \alpha u \quad (x,t) \in \Omega \times R_+ \tag{1.1}$$

$$\frac{\partial v}{\partial t} - b\Delta v = f(u,v) - \sigma\kappa(v) \quad (x,t) \in \Omega \times R_+ \tag{1.2}$$

with the boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{on } \partial\Omega \times R_+, \tag{1.3}$$

and the initial data

$$u(0,x) = u_0(x) \geq 0; \quad v(0,x) = v_0(x) \geq 0 \quad \text{in } \Omega, \tag{1.4}$$

where Ω is a smooth open bounded domain in R^n, with boundary $\partial\Omega$ of class C^1 and η is the outer normal to $\partial\Omega$. The constants of diffusion a, b are positive and such that $a \neq b$ and Π, α, σ are positive constants, κ and f are nonnegative functions of class $C^1(R_+)$ and $C^1(R_+ \times R_+)$ respectively.

The reaction-diffusion system (1.1) − (1.4) arises in the study of physical, chemical, and various biological processes including population dynamics (especially AIDS, see C. Castillo-Chavez et al. [3], for further details see [5, 7, 12, 16, 17]).

The case $\Pi = 0$, $\alpha = 0, \sigma = 0$ and $f(u,v) = h(u)T(v)$, with $h(u) = u$ (for simplicity), has been studied by many authors. Alikakos [1] established the existence of global solutions when $T(v) \leq C(1 + |v|^{(n+2)/n})$. Then Massuda [13] obtained a positive result for the case $T(v) \leq C(1 + |v|^{\alpha})$ with arbitrary $\alpha > 0$. The question when $T(v) = e^{\alpha v^{\beta}}$, $0 < \beta < 1$, $\alpha > 0$ was positively answered by Haraux and Youkana [9], using Lyapunov function techniques, see also Barabanova [2] for $\beta = 1$, with some conditions and later on by Kanel and Kirane [11], using useful properties inherent to the Green function. The idea behind the Lyapunov functional stems from Zelenyak's article [18], which has also been used by Crandall et al. [4] for other purposes.

The goal of this work is to generalize the existing results of L. Melkemi et al. [14],

2000 *Mathematics Subject Classification.* 35K57, 35K45.
Key words and phrases. Reaction-diffusion systems; Lyapunov functional; Global solution.

where they established the existence of global solutions, when $f(\xi,\tau) \leq \psi(\xi)\varphi(\tau)$ such that

$$\lim_{\tau\to+\infty} \frac{\ln(1+\varphi(\tau))}{\tau} = 0.$$

Hence, the main purpose of this paper is to give a positive answer, concerning the global existence and the uniform boundedness in time, of solutions of system $(1.1)-(1.4)$, with exponential nonlinearity, such that f satisfies

(A1) $\forall\tau \geq 0,\ f(0,\tau) = 0,$
(A2) $\forall\xi \geq 0,\ \forall\tau \geq 0,\ 0 \leq f(\xi,\tau) \leq \varphi(\xi)(\tau+1)^{\lambda}e^{r\tau},$
(A3) $\kappa(\tau) = \tau^{\mu},\ \mu \geq 1,$

where r, λ are positive constants, such that $\lambda \geq 1$, φ is a nonnegative function of class $C(R^{+})$.
Our aim in this work, is to establish the global existence of solutions of $(1.1)-(1.4)$, with exponential nonlinearity expressed by the condition (A2), for arbitrary v_0 and u_0 satisfying

$$\max\left(\| u_0 \|_{\infty}, \frac{\Pi}{\alpha}\right) < \frac{\theta^2}{2-\theta} \quad \frac{8ab}{rn(a-b)^2}, \tag{1.5}$$

where $\theta < 1$ is a positive real number very close to 1.
For this end we use comparison principle and Lyapunov function techniques.

2. Existence of local solutions

The usual norms in spaces $L^p(\Omega)$, $L^{\infty}(\Omega)$ and $C(\overline{\Omega})$ are respectively denoted by

$$\| u \|_p^p = \frac{1}{|\Omega|}\int_{\Omega} | u(x) |^p \, dx, \quad \| u \|_{\infty} = \max_{x\in\Omega} | u(x) | .$$

Concerning a local existence, we can conclude directly from the theory of abstract semilinear equations (see A. Friedman [6], D. Henry [10], A. Pazy [15]), that for nonnegative functions u_0 and v_0 in $L^{\infty}(\Omega)$, there exists a unique local nonnegative solution (u,v) of system $(1.1)-(1.4)$ in $C(\overline{\Omega})$ on $]0,T^*[$, where T^* is the eventual blowing-up time.

3. Existence of global solutions

Using the comparison principle, one obtains

$$0 \leq u(t,x) \leq \max(\| u_0 \|_{\infty}, \frac{\Pi}{\alpha}), \tag{3.1}$$

from which it remains to establish the uniform boundedness of v.

According to the results of [8], it is enough to show that

$$\| f(u,v) - \sigma\kappa(v) \|_p \leq C \tag{3.2}$$

(where C is a nonnegative constant independent of t) for some $p > \frac{n}{2}$.

The main result of this paper is

Theorem 3.1. *Under the assumptions* $(A1)-(A3)$ *and* (1.5), *the solutions of* $(1.1)-(1.4)$ *are global and uniformly bounded on* $[0,+\infty[$.

Let be ω, β, γ and M positive constants such that $\omega \geq 1$,

$$\beta = \theta \frac{4ab}{(a-b)^2}, \quad \gamma = \max\big(\lambda, \mu, \frac{(\beta+1)(2-\theta)Mr}{\beta\theta(1-\theta)}\big) \tag{3.3}$$

and

$$M = \max\big(\parallel u_0 \parallel_\infty, \frac{\Pi}{\alpha}\big) < \frac{\theta^2}{2-\theta} \quad \frac{8ab}{rn(a-b)^2}. \tag{3.4}$$

We can choose

$$p = \frac{\theta^2}{2-\theta} \frac{4ab}{(a-b)^2 Mr} \tag{3.5}$$

as consequence of (3.4), we observe that $p > \frac{n}{2}$.
The key result needed to prove the theorem 3.1 is the following

Proposition 3.2. *Assume that* $(A1)-(A3)$ *hold and let* (u,v) *be a solution of* $(1.1)-(1.4)$ *on* $]0, T^*[$, *with arbitrary* v_0 *and* u_0 *satisfying* (1.5). *Let*

$$R_\rho(t) = \rho \int_\Omega u dx + \int_\Omega \left(\frac{M}{(2-\theta)M - u} \right)^\beta (v+\omega)^{\gamma p} e^{prv} dx. \tag{3.6}$$

Then, there exist $p > n/2$ *and positive constants* s *and* Γ *such that*

$$\frac{dR_\rho}{dt} \leq -sR_\rho + \Gamma. \tag{3.7}$$

It's very important to state a number of lemmas, before proving this proposition.

Lemma 3.3. *If* (u,v) *is a solution of* $(1.1)-(1.4)$ *then*

$$\int_\Omega f(u,v) dx \leq \Pi|\Omega| - \frac{d}{dt} \int_\Omega u(t,x) dx. \tag{3.8}$$

Proof. We integrate both sides of (1.1),

$$f(u,v) = \Pi - \alpha u - \frac{d}{dt} u(t,x) - a\Delta u$$

satisfied by u, which is positive and then we find (3.8). □

Lemma 3.4. *Let be* ψ *a nonnegative function of class* $C(R^+)$, *such that*

$$\lim_{\tau \to +\infty} \frac{\psi(\tau)}{\tau + \omega} = 0$$

and let A *be positive constant. Then there exists* $N_1 > 0$, *such that*

$$[\frac{\psi(\tau)}{\tau+\omega} - A](\tau+\omega)^{\gamma p} e^{pr\tau} f(\xi,\tau) \leq N_1 f(\xi,\tau), \tag{3.9}$$

for all $0 \leq \xi \leq M$ *and* $\tau \geq 0$.

Proof. Since

$$\lim_{\tau \to +\infty} \frac{\psi(\tau)}{\tau + \omega} = 0,$$

there exists $\tau_0 > 0$, such that for all $0 \leq \xi \leq K, \tau > \tau_0$, we have

$$[\frac{\psi(\tau)}{\tau+\omega} - A](\tau+\omega)^{\gamma p} e^{pr\tau} f(\xi,\tau) \leq 0.$$

Now if τ is in the compact interval $[0, \tau_0]$, then the continuous function

$$\chi(\xi, \tau) = [\psi(\tau)(\tau+\omega)^{\gamma p-1} - A(\tau+\omega)^{\gamma p}]e^{pr\tau}$$

is bounded. $\square$

Lemma 3.5. *For all $\tau \geq 0$ we have*

$$[\frac{\Pi\beta}{(1-\theta)M} - \sigma p\kappa(\tau)(\frac{\gamma}{\tau+\omega} + r)](\tau+\omega)^{\gamma p}e^{pr\tau} \leq -s(\tau+\omega)^{\gamma p}e^{pr\tau} + B_1, \quad (3.10)$$

where B_1 and s are positive constants.

Proof. Let us put

$$\xi = \frac{\Pi\beta}{(1-\theta)M} + s$$

$$\frac{\Pi\beta}{(1-\theta)M}(\tau+\omega)^{p\gamma}e^{pr\tau} - \sigma p\kappa(\tau)[\gamma(\tau+\omega)^{\gamma p-1} + r(\tau+\omega)^{\gamma p}]e^{pr\tau} =$$

$$\left(\frac{\Pi\beta}{(1-\theta)M} - \xi\right)(\tau+\omega)^{p\gamma}e^{pr\tau} + \left(\frac{\xi}{\kappa(\tau)} - \sigma rp\right)\kappa(\tau)(\tau+\omega)^{\gamma p}e^{pr\tau},$$

then, using Lemma 3.4 we can conclude the result. $\square$

Proof. (of Proposition 3.2)
Let

$$g(u) = \left(\frac{M}{(2-\theta)M - u}\right)^{\beta},$$

so that

$$R_\rho(t) = \rho\int_\Omega u dx + G(t),$$

where

$$G(t) = \int_\Omega g(u)(v+\omega)^{\gamma p}e^{prv}dx.$$

Differentiating G with respect to t and a simple use of Green's formula gives

$$G'(t) = I + J,$$

where

$$\begin{aligned} I = \quad & -a\int_\Omega g''(u)(v+\omega)^{\gamma p}e^{prv}|\nabla u|^2dx \\ & -(a+b)\int_\Omega g'(u)[\gamma p(v+\omega)^{\gamma p-1} + pr(v+\omega)^{\gamma p}]e^{prv}\nabla u\nabla v dx \\ & -b\int_\Omega g(u)[\gamma p(\gamma p-1)(v+\omega)^{\gamma p-2} + 2\gamma p^2 r(v+\omega)^{\gamma p-1} + p^2r^2(v+\omega)^{\gamma p}]e^{prv}|\nabla v|^2dx, \end{aligned}$$

$$\begin{aligned} J = \quad & \int_\Omega \Pi g'(u)(v+\omega)^{\gamma p}e^{prv}dx - \int_\Omega \alpha g'(u)u(v+\omega)^{\gamma p}e^{prv}dx \\ & + \int_\Omega \left(g(u)\left[\gamma p(v+\omega)^{\gamma p-1} + rp(v+\omega)^{\gamma p}\right] - g'(u)(v+\omega)^{\gamma p}\right)f(u,v)e^{prv}dx \\ & - \int_\Omega \sigma[\gamma p(v+\omega)^{\gamma p-1} + rp(v+\omega)^{\gamma p}]\kappa(v)g(u)e^{prv}dx. \end{aligned}$$

We can see that I involves a quadratic form with respect to ∇u and ∇v, which is nonnegative if

$$\begin{aligned}\delta = \ & \big(p(a+b)g'(u)[\gamma(v+\omega)^{\gamma p-1}+r(v+\omega)^{\gamma p}]\big)^2\\ & -4ab\gamma p(\gamma p-1)g''(u)g(u)(v+\omega)^{2\gamma p-2}\\ & -4abg''(u)g(u)(v+\omega)^{\gamma p}[2\gamma p^2 r(v+\omega)^{\gamma p-1}+p^2r^2(v+\omega)^{\gamma p}]\leq 0.\end{aligned}$$

Indeed

$$\begin{aligned}\delta = \ & [(p\gamma)^2(a+b)^2\beta^2-4ab\beta(\beta+1)p\gamma(p\gamma-1)]\frac{g(u)^2(v+\omega)^{2p\gamma-2}}{((2-\theta)M-u)^2}\\ & +[(a+b)^2\beta^2-4ab\beta(\beta+1)]\frac{rp^2g(u)^2(v+\omega)^{2p\gamma-1}}{((2-\theta)M-u)^2}[2\gamma+r(v+\omega)],\end{aligned}$$

the choice of β and γ gives

$$\begin{aligned}\delta \leq \ & [\beta+1-p\gamma(1-\theta)]\frac{4ab\beta p\gamma g(u)^2(v+\omega)^{2p\gamma-2}}{((2-\theta)M-u)^2}\\ & +4ab(\theta-1)\frac{rp\beta g(u)^2(v+\omega)^{2p\gamma-1}}{((2-\theta)M-u)^2}[2+(rp)(v+\omega)]\leq 0,\end{aligned}$$

it follows that

$$I\leq 0.$$

Concerning the second term J, we can observe that

$$\begin{aligned}J \leq \ & \int_\Omega\left(\frac{\Pi\beta}{(1-\theta)M}-\sigma p\kappa(v)[\frac{\gamma}{v+\omega}+r]\right)g(u)(v+\omega)^{p\gamma}e^{prv}dx\\ & +\int_\Omega\left(p[\frac{\gamma}{v+\omega}+r]-\frac{\beta}{(2-\theta)M-u}\right)f(u,v)g(u)(v+\omega)^{\gamma p}e^{prv}dx.\end{aligned}$$

Using Lemma 3.5, we get

$$\begin{aligned}J \leq \ & \int_\Omega[-s(v+\omega)^{p\gamma}e^{prv}+B_1]g(u)dx\\ & +\int_\Omega\left(p[\frac{\gamma}{v+\omega}+r]-\frac{\theta}{2-\theta}\frac{4ab}{(a-b)^2M}\right)f(u,v)g(u)(v+\omega)^{\gamma p}e^{prv}dx,\end{aligned}$$

or

$$\begin{aligned}J \leq \ & \int_\Omega[-s(v+\omega)^{p\gamma}e^{prv}+B_1]g(u)dx\\ & +\int_\Omega\left(\frac{p\gamma}{v+\omega}-\frac{\theta(1-\theta)}{2-\theta}\frac{4ab}{(a-b)^2M}\right)f(u,v)g(u)(v+\omega)^{\gamma p}e^{prv}dx\\ & +\int_\Omega\left(pr-\frac{\theta^2}{2-\theta}\frac{4ab}{(a-b)^2M}\right)f(u,v)g(u)(v+\omega)^{\gamma p}e^{prv}dx.\end{aligned}$$

From Lemma 3.4 and formula (3.5), it follows

$$\begin{aligned}J \leq \ & \int_\Omega[-s(v+\omega)^{p\gamma}e^{prv}+B_1]g(u)dx\\ & +N_1\int_\Omega f(u,v)g(u)dx.\end{aligned}$$

In addition

$$g(u) \leq \left(\frac{1}{1-\theta}\right)^{\beta},$$

then

$$J \quad \leq \quad -sG(t)+ \mid \Omega \mid B_1\left(\frac{1}{1-\theta}\right)^{\beta} + N_1\left(\frac{1}{1-\theta}\right)^{\beta}\int_{\Omega} f(u,v)dx.$$

Then if we put

$$B = B_1 \mid \Omega \mid \left(\frac{1}{1-\theta}\right)^{\beta}$$

and

$$\rho = N_1\left(\frac{1}{1-\theta}\right)^{\beta}.$$

Then, if we use Lemma 3.3,

$$\begin{aligned} J \leq & -sR_{\rho}(t) + s\rho\int_{\Omega} u(t,x)dx + B + \rho\Pi \mid \Omega \mid -\rho\frac{d}{dt}\int_{\Omega} u(t,x)dx \\ \leq & -sR_{\rho}(t) + [sM+\Pi]\rho \mid \Omega \mid +B - \rho\frac{d}{dt}\int_{\Omega} u(t,x)dx, \end{aligned}$$

it follows that

$$\frac{dR_{\rho}}{dt} \leq -sR_{\rho} + \Gamma,$$

where $\Gamma = [sM+\Pi]\rho \mid \Omega \mid +B$. □

Proof. (of Theorem 3.1)
Multiplying (3.7) by e^{st} and integrating the inequality, it implies the existence of a positive constant $C > 0$ independent of t such that

$$R_{\rho}(t) \leq C.$$

Since

$$g(u) \geq (\frac{1}{2-\theta})^{\beta},$$

$$\begin{aligned} \int_{\Omega} (v+\omega)^{\gamma p} e^{prv} dx \quad & \leq \quad (2-\theta)^{\beta} R_{\rho}(t) \\ & \leq \quad C(2-\theta)^{\beta}. \end{aligned}$$

Since $\omega \geq 1$ and (3.3) we have also,

$$\int_{\Omega} (v+1)^{\lambda p} e^{prv} dx \leq \int_{\Omega} (v+\omega)^{\gamma p} e^{prv} dx \leq C(2-\theta)^{\beta},$$

$$\int_{\Omega} v^{\mu p} dx \leq \int_{\Omega} (v+\omega)^{\gamma p} dx \leq C(2-\theta)^{\beta}.$$

We put

$$A = \max_{0 \leq \xi \leq M} \varphi(\xi),$$

according to $(A1) - (A3)$, we have

$$\int_{\Omega} f(u,v)^p dx \leq \int_{\Omega} A^p (v+1)^{\lambda p} e^{prv} dx \leq A^p C(2-\theta)^{\beta} = A^p H^p,$$

we conclude

$$\|f(u,v)-\sigma\kappa(v)\|_p \leq \|f(u,v)\|_p + \|\sigma\kappa(v)\|_p \leq H(A+\sigma).$$

By the preliminary remarks (introduction of section 3), we conclude that the solution of $(1.1)-(1.4)$ is global and uniformly bounded on $[0,+\infty[\times\Omega$.

□

Acknowledgments. I want to thank the anonymous referee for his/her fruitful comments that improved the final version of this article.

References

[1] N. Alikakos, *L^p-Bounds of solutions of reaction-diffusion equations*, comm. Partial Differential Equations. 4 (1979), 827-868.

[2] A. Barabanova,*On the global existence of solutions of reaction-diffusion equation with exponential nonlinearity*, Proc. Amer. Math. Soc. 122 (1994), 827-831.

[3] C. Castillo-Chavez, K. Cooke, W. Huang and S. A. Levin, *On the role of long incubation periods in the dynamics of acquires immunodeficiency syndrome, AIDS*, J. Math. Biol. 27(1989) 373-398.

[4] M. Crandall, A. Pazy, and L. Tartar, *Global existence and boundedness in reaction-diffusion systems*. SIAM J. Math. Anal. 18 (1987), 744-761.

[5] E. L. Cussler, *Diffusion*, Cambridge University Press, second edition, 1997.

[6] A. Friedman, *Partial differential equation of parabolic type*, Prentice Hall, 1964.

[7] Y. Hamaya, *On the asymptotic behavior of a diffusive epidemic model (AIDS)*, Nonlinear Analysis, 36 (1999), 685-696.

[8] A. Haraux and M. Kirane, *Estimation C^1 pour des problemes paraboliques semi−lineaires*. Ann. Fac Sci. Toulouse Math. 5 (1983), 265-280.

[9] A. Haraux and A. Youkana, *On a result of K. Masuda concerning reaction-diffusion equations*. Tohoku Math. J. 40 (1988), 159-163.

[10] D. Henry, *Geometric theory of semilinear parabolic equations*. Lecture Notes in Mathematics 840, Springer−Verlag, New york, 1981.

[11] J. I. Kanel and M. Kirane, *Global solutions of reaction-diffusion systems with a balance law and nonlinearities of exponential growth*. Journal of Differential Equations, 165, 24-41 (2000).

[12] M. Kirane, *Global bounds and asymptotics for a system of reaction−diffusion equations*. J. Math Anal. Appl. 138 (1989), 1172-1189.

[13] K. Masuda, *On the global existence and asymptotic behaviour of reaction-diffusion Equations*. Hokkaido Math. J. 12 (1983), 360-370.

[14] L. Melkemi, A. Z. Mokrane, and A. Youkana, *Boundedness and large-time behavior results for a diffusive epedimic model*. J. Applied Math, volume 2007, 1-15.

[15] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*. Springer Verlag, New York, 1983.

[16] G. F. Webb, *A reaction-diffusion model for a deterministic diffusive epidemic*, J. Math anal. Appl. 84 (1981), 150-161.

[17] E. Zeidler, *Nonlinear functional analysis and its applications*, Tome II/b, Springer Verlag, 1990.

[18] T. I. Zelenyak, *Stabilization of solutions to boundary value problems for second order parabolic equations in one space variable*, Differentsial'nye Uravneniya 4 (1968), 34-45.

(Received August 31, 2009)

El Hachemi Daddiouaissa
Department of Mathematics University Kasdi Merbah, UKM Ouargla 30000, Algeria.
E-mail address: dmhbsdj@gmail.com